

About Cartan-Subalgebras in Lie-Algebras associated to Associative Algebras

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Abstract

Every associative K -algebra A is with respect to the multiplication $a \circ b := ab - ba$ for all $a, b \in A$ a Lie-Algebra A° , also known as the Lie-Algebra associated with A . In [8] S. Siciliano studies Cartan-Subalgebras of A° . These are nilpotent subalgebras C of A° which coincide with the normalizer $N_{A^\circ}(C)$ of C in A .

For every finite-dimensional associative unitary K -algebra A Siciliano proves that the Cartan-Subalgebras of A° are exactly the centralizers of the maximal tori of A . Especially, Cartan-Subalgebras of A° are subalgebras of the associative algebra A . A torus is a commutative unitary subalgebra of A for which every element is separable over K . An element is separable if its minimal polynomial is a product of pairwise irreducible separable polynomials of $K[t]$. For the algebra classes "finite-dimensional central division algebras" and "finite-dimensional soluble algebras" S.Siciliano describes the Cartan-Subalgebras in terms of 'maximal separable subfields' and 'radical complements'.

At first we give alternative proves for Sicilianos results concerning these two algebra classes. As an example we compute the Cartan-Subalgebras for soluble group algebras and especially for soluble group algebras related to dihedral groups.

After this we describe the Cartan-Subalgebras for finite-dimensional associative division, simple, semi-simple and reduced algebras.

As a associative subalgebra we analyze the associative structure of Cartan-Subalgebras. This investigation is closely connected to the question for which finite-dimensional associative unitary algebras its group of units is nilpotent.

We close this article by giving a strategy for computing Cartan-Subalgebras for associative algebras with separable radical complements. An easy consequence is again the description of the Cartan-Subalgebras in the case of

soluble algebras. We demonstrate this strategy for group algebras related to dihedral groups.

Cartan-Subalgebras are associative subalgebras

S.Siciliano especially proves (see [8], theorem 1) that for every finite-dimensional associative unitary K -algebra every Cartan-Subalgebra of its associated Lie-Algebra is an associative subalgebra. We generalize this statement to arbitrary associative algebras.

Definition 1

- (i) For all $n \in \mathbb{N}$ we set $\underline{n} := \mathbb{N}_{\leq n}$.
- (ii) For all $n \in \mathbb{N}_{\geq 2}$ we define
 $T_n := \{(\alpha, \beta) \mid \exists r \in \underline{n-1}, a_1, \dots, a_n \in \underline{n} : \alpha = (a_1, \dots, a_r),$
 $\beta = (a_{r+1}, \dots, a_n), a_1 < \dots < a_r, a_{r+1} < \dots < a_n, \underline{n} = \{a_1, \dots, a_n\}\}.$
- (iii) For every Lie-Algebra L and $l \in L$ let $ad(l)$ be the adjoint representation of l with respect to L .
- (iv) For every nilpotent Lie-algebra L let $cl(L)$ be the nilpotency class of L .

Remark 1

Let A be an associative K -algebra and $a, b \in A$.

- (i) For all $h_1 \in A$ the derivation rule
 $(ab)ad(h_1) = (a ad(h_1))b + a(b ad(h_1))$ is valid.
- (ii) For all $n \in \mathbb{N}_{\geq 2}, h_1, \dots, h_n \in A$ we get by (i) and induction

$$\begin{aligned} (ab)ad(h_1)\dots ad(h_n) &= (a ad(h_1)\dots ad(h_n))b + a(b ad(h_1)\dots ad(h_n)) \\ &+ \sum_{((\alpha_1, \dots, \alpha_r), (\alpha_{r+1}, \dots, \alpha_n)) \in T_n} (a ad(h_{\alpha_1})\dots ad(h_{\alpha_r}))(b ad(h_{\alpha_{r+1}})\dots ad(h_{\alpha_n})). \diamond \end{aligned}$$

Proposition 1

Let A be an associative (unitary) K -algebra and C be a Cartan-Subalgebra of A° . Then C is an (unitary) associative subalgebra of A .

Proof: As a subalgebra of A° the set C is a K -subspace of A . Let $a, b \in C$. Then we have to prove $ab \in C = N_{A^\circ}(C)$. This is equivalent to $(ab)ad(h_1) \in C = N_{A^\circ}(C)$ for all $h_1 \in C$. By induction, we must show that there exists a $n \in \mathbb{N}$ such that for all $h_1, \dots, h_n \in C$ the statement $(ab)ad(h_1)\dots ad(h_n) \in C$

is valid. Let $n := 2 \cdot cl(C)$. Using part (ii) of remark 1 the statement $(ab)ad(h_1)...ad(h_n) = 0 \in C$ is valid (At least one factor of every summand in the sum of part (ii) in remark 1 is equal to zero.). Is A unitary then $1_A \in N_{A^\circ}(C) = C$. \diamond

Division algebras

S. Siciliano proves by theorem 1 in [8] that the separable maximal subfields of a finite-dimensional associative central K -division algebra are exactly the Cartan-Subalgebras of its associated Lie-algebra. Here we give an alternative proof of his theorem and describe the Cartan-Subalgebras for not necessary central division algebras.

Remark 2

Let A, B be associative K -algebras.

- (i) For all $a_1, a_2 \in A, b_1, b_2 \in B$ the equation $(a_1 \otimes b_1) \circ (a_2 \otimes b_2) = (a_1 \circ a_2) \otimes (b_1 b_2) + (a_1 a_2) \otimes (b_1 \circ b_2)$ is valid.
- (ii) Let A° be nilpotent and B° be abelian. By (i) and induction the algebra $(A \otimes B)^\circ$ is nilpotent with the same nilpotency class as A° . \diamond

Remark 3

Let K be a field and $n \in \mathbb{N}$ with $n \geq 2$. Then $gl(n, K)$ is not nilpotent. \diamond

Let A be an associative K -algebra and T, S be subsets of A . Then we call $C_A(T)$ the centralizer of T in A , define $C_S(T) := C_A(T) \cap S$ and call $Z(A) := C_A(A)$ the center of A . Additionally for all $n \in \mathbb{N}$ we call $A^{n \times n}$ the algebra of all $n \times n$ -matrices over A . For a central-simple finite-dimensional associative K -Algebra let $ind(D)(= ind_K(A))$ be the index of D .

Proposition 2

Let D be a finite-dimensional associative non-commutative K -division algebra. Then D° is not nilpotent.

Proof: D is as $Z(D)$ -algebra a central division algebra, and D° as K -algebra nilpotent if and only if D° is nilpotent as $Z(D)$ -algebra. Therefor we assume that D is central.

Let T be a maximal subfield of D° , and we assume that D° is nilpotent. Then $(D \otimes T)^\circ$ is nilpotent by remark 2. It is well-known (see e.g. [7]) that $D \otimes T$ and $T^{ind(D) \times ind(D)}$ are isomorphic. By using remark 3 we get $n = 1$. \diamond

Now we can prove the following enhancement of theorem 2 in [8] and of a theorem by E. Noether:

Theorem 1

Let D be a finite-dimensional associative central K -division algebra.

- (i) The maximal separable subfields of D are exactly the separable maximal subfields of D .
- (ii) There exists a separable maximal subfield. (Noether)
- (iii) The Cartan-Subalgebras of D° are exactly the separable maximal subfields of D . (Siciliano)

Proof: ad(i): Let T be a maximal separable subfield of D . Then T is a maximal torus of D (Every unitary subalgebra of D is a division algebra and tori are commutative.). By Theorem 1 in [8] the algebra $C_D(T)$ is a Cartan-Subalgebra of D° . As D is a finite-dimensional associative K -division algebra so is $C_D(T)$ as well, and by Proposition 2 the algebra $C_D(T)$ is a subfield of D . $C_D(T)$ is maximal Lie-nilpotent so that $C_D(T)$ is a maximal subfield of D . Maximal subfields are self-centralizing (see for instance [7]) so that $C_D(C_D(T)) = C_D(T)$ is valid. By the double-centralizer-theorem we get $C_D(C_D(T)) = T$. Thus $T = C_D(T)$ is a separable maximal subfield of D . Obviously every separable maximal subfield is a maximal separable subfield of D .

ad(ii): $K \cdot 1_D$ is a separable subfield of D . Thus there exists a maximal separable subfield of D which is by (i) a separable maximal subfield of D .

ad(iii): By theorem 1 in [8] the Cartan-Subalgebras of D° are exactly the centralizers of the maximal tori of D . A maximal torus of D is a maximal separable subfield of D (Every unitary subalgebra of D is a K -division subalgebra of D and a torus is commutative.). By (i) we get that the Cartan-Subalgebras of D° are exactly the centralizers of the separable maximal subfields of D . Now the proof is finished because (see for instance [7]) every maximal subfield of D is self-centralizing. \diamond

A consequence of theorem 1 is:

Corollary 1

Let D be a finite-dimensional associative central K -division algebra.

- (i) The Cartan-Subalgebras of D° are Lie-isomorphic and $\text{ind}(D)$ -dimensional.

- (ii) For a perfect field K all Cartan-Subalgebras of D° are exactly the maximal subfields of D . \diamond

We extend theorem 1 to finite-dimensional associative and not necessary central K -division algebras:

Theorem 2

Let D be a finite-dimensional associative K -division algebra.

- (i) The Cartan-Subalgebras of D° are exactly the maximal subfields of D which are separable over $Z(D)$.
- (ii) There exists a maximal subfield of D which is separable over $Z(D)$.
- (iii) The maximal subfields of D which are separable over $Z(D)$ are exactly those subfields of D which are maximal separable over $Z(D)$.

Proof: The proof is an consequence of theorem 1 by regarding the following facts:

1. Every maximal subfield and every Cartan-Subalgebra of D contains the center of D .
2. D is central as $Z(D)$ -Algebra.
3. The Cartan-Subalgebras of D° as K -and $Z(D)$ -Lie-Algebra are the same. \diamond

An easy consequence of theorem 2 is:

Corollary 2

Let D be a finite-dimensional associative K -division algebra.

- (i) All Cartan-Subalgebras of D° are Lie-isomorphic and $\text{ind}_{Z(D)}(D) \cdot \dim_K(Z(D))$ -dimensional.
- (ii) If $Z(D)$ is separable over K then the Cartan-Subalgebras of D° are exactly the separable maximal subfields of D .
- (iii) For a perfect field K the Cartan-Subalgebras of D° are exactly the maximal subfields of D . \diamond

We close this section by giving the following dimension-formula related to this corollary: $\text{ind}_K(D)^2 = \dim_K(D) = \dim_K(Z(D)) \cdot \dim_{Z(D)}(D) = \dim_K(Z(D)) \cdot \text{ind}_{Z(D)}(D)^2$.

Soluable Algebras

In [1] and [8] T. Bauer and S. Siciliano prove for a finite-dimensional associative unitary soluble K -Algebra A with separable radical factor algebra that the Cartan-Subalgebras of A° are exactly the centralizers of those subalgebras which are direct to the radical – known as radical complements. We will prove this theorem in a different way and analyze Cartan-Subalgebras of soluble group algebras.

Let A be an associative K -algebra over a field K , a be an algebraic element and T be a subset of A . By $\min_{a,K}$ we denote the minimal polynomial of a over K . Furthermore, we denote by $\text{char}(K)$ the characteristic of K , by $\langle T \rangle_K$ and $K[T]$ respectively the K -generating and algebra-generating system respectively of T in A .

Our analysis is based on the following lemma (see for instance theorem 5.3.1 in [9]):

Lemma 1

Let A be a finite-dimensional associative commutative unitary K -algebra. A is separable if and only if every element of A is separable over K . \diamond

Let A be an associative K -algebra. By $\text{rad}(A)$ we denote the nil radical of A .

Theorem 3

Let A be a finite-dimensional associative unitary soluble K -algebra with separable radical complement. The maximal tori of A are exactly the radical complements of A .

Proof: ' \rightarrow :' Let T be a radical complement of A . Then T is a commutative separable unitary subalgebra of A . Using lemma 1 we conclude that T is a torus of A . Let S be a torus von A which includes T . Again by lemma 1 the algebra S is a separable K -subalgebra of A which is direct to $\text{rad}(A)$. Calculating the dimensions we conclude $T = S$.

' \leftarrow :' Let T be a maximal torus of A . T is – as a consequence of lemma 1 – a separable subalgebra of A . By an enhancement of the Wedderburn-Malcev-Conjugacy-Theorem (see e.g. corollary 2.3.7 in [9]) T is a subalgebra of a radical complement of A . The proof is completed using ' \rightarrow '. \diamond

By theorem 3, theorem 1 in [8] and the Wedderburn-Malcev-Theorem we get:

Theorem 4 (Bauer)

Let A be a finite-dimensional associative unitary soluble K -algebra with separable radical factor algebra.

- (i) The Cartan-Subalgebras of A° are exactly the centralizers of the radical complements of A .
- (ii) The Cartan-Subalgebras of A° are conjugated with respect to the normal subgroup $1_A + \text{rad}(A)$ of its the group of units. \diamond

Cartan-Subalgebras of soluble Group Algebras

Let K be a field and G be a finite group. From 3.2.20 in [9] we conclude that KG is soluble if and only if G is abelian or $\text{char}(K) = p$ and G' – the derivation of G – is a p -group. For abelian G clearly $(KG)^\circ$ is nilpotent. Let $\text{char}(K) = p$ and G' be a p -group. As G' is a normal p -subgroup of G we conclude by Sylows-Theorems that G has exactly one (normal) p -Sylowsubgroup P . By the Schur-Zassenhaus-Theorem there exists a complement H of P in G . Let α be the linearization of the canonical group-epimorphism from G onto the factor group G/P . Then the kernel is given by $\text{Kern } \alpha = KG \text{Aug}(KP) = \text{Aug}(KP)KG$ ($\text{Aug}(KP)$ is the augmentation ideal of KP). By a theorem of Wallace $\text{Aug}(KP)$ is nilpotent and hence $\text{Kern } \alpha$ is nilpotent as well. The factor algebra KG modulo $\text{Kern } \alpha$ is isomorphic to $K(G/P)$ and hence isomorphic to KH as well. By Maschke's theorem KH is semi-simple and hence separable using 1.9.4 in [9]. We conclude that $\text{rad}(KG) = KG \text{Aug}(KP)$ is valid and KH is a separable radical complement of KG .

By Theorem 4 all Cartan-Subalgebras of $(KG)^\circ$ are conjugated by $1 + \text{rad}(KG)$ to $C_{KG}(KH) = C_{\text{rad}(KG)}(KH) \oplus KH$.

Using standard linear algebra we conclude that the set $\{(a-1)h \mid 1 \neq a \in P, h \in H\}$ is a K -basis of $\text{rad}(KG)$ which is useful for the determination of $C_{\text{rad}(KG)}(KH)$. \diamond

Let G be a group, T be a subset of G and a be an element of finite order of G . By $\langle T \rangle$ we denote the subgroup of G generated by T and by $o(a)$ the order of a in G .

Cartan-Subalgebras of soluble Group Algebras over Dihedral Groups

- (i) Let G be a group, $n \in \mathbb{N}$ and $a, b \in G$ such that $o(a) = n$, $o(b) = 2$, $G = \langle a, b \rangle$ and $a^b = a^{-1}$ are valid. If n is not divisible by 2 the derivation of G is given by $G' = \langle a \rangle$. In the other case the equation $G' = \langle a^2 \rangle$ is true.

(ii) Let K be a field. By 3.2.20 in [9] we conclude that KG is soluble if and only if G is abelian or $\text{char}(K) = p$ and G' is a p -group. By (i) the following cases are to be analyzed for the determination of Cartan-Subalgebras (p is a prime not equal to 2):

- (a) G is abelian.
- (b) G is a 2-group and $\text{char}(K) = 2$.
- (c) n is a power of p and $\text{char}(K) = p$.
- (d) $\frac{n}{2}$ is a power of p and $\text{char}(K) = p$.

Now we describe the Cartan-Subalgebras of $(KG)^\circ$ for this four cases and use the conclusions of the section **Cartan-Subalgebras of soluble Group Algebras** for our analysis.

(iii)(a) G is abelian only for $n \in \underline{2}$. Then $(KG)^\circ$ is nilpotent.

(iii)(b) Let G be a 2-group and $\text{char}(K) = 2$. Then $KG = \text{Aug}(KG) \oplus K \cdot 1_G$ holds and $(KG)^\circ$ is nilpotent.

(iii)(c) Let n be a power of p and $\text{char}(K) = p$. $G' = \langle a \rangle$ is the p -Sylowsubgroup of G with complement $\langle b \rangle$. $K\langle b \rangle$ is a radical complement of KG with K -basis $\{1, b\}$, and the set $\{a^s - 1, (a^s - 1)b \mid s \in \underline{n-1}\}$ is a K -basis of the radical. The Cartan-Subalgebras are conjugated by $1 + \text{rad}(KG)$ to $C_{KG}(K\langle b \rangle) = C_{\text{rad}(KG)}(K\langle b \rangle) \oplus K\langle b \rangle$. We calculate the centralizer of $K\langle b \rangle$ in $\text{rad}(KG)$ and show that its dimension is $n - 1$. Hence all Cartan-Subalgebras of $(KG)^\circ$ are $(n + 1)$ -dimensional.

Let $x \in \text{rad}(KG)$ like $x = \sum_{i=1}^{n-1} k_i(a^i - 1) + \sum_{j=1}^{n-1} l_j(a^j - 1)b$. We calculate:

$$\begin{aligned}
 & \forall y \in K\langle b \rangle : xy = yx \iff \\
 & \sum_{i=1}^{n-1} k_i a^i b + \sum_{j=1}^{n-1} l_j a^j - \sum_{i=1}^{n-1} k_i b a^i - \sum_{j=1}^{n-1} l_j b a^j b = 0 \iff \\
 & \sum_{i=1}^{n-1} k_i (a^i b - a^{-i} b) + \sum_{j=1}^{n-1} l_j (a^j - a^{-j}) = 0 \iff \\
 & \sum_{i=1}^{n-1} k_i (a^i b - a^{n-i} b) + \sum_{j=1}^{n-1} l_j (a^j - a^{n-j}) = 0 \iff \\
 & \sum_{i=1}^{\frac{n-1}{2}} (k_i - k_{n-i}) a^i b + \sum_{j=1}^{\frac{n-1}{2}} (l_j - l_{n-j}) a^j = 0 \iff \\
 & \forall i, j \in \underline{\frac{n-1}{2}} : k_i = k_{n-i} \wedge l_j = l_{n-j}
 \end{aligned}$$

(iii)(d) Let $\frac{n}{2}$ be a power of p like $n = 2 \cdot p^r$ and $\text{char}(K) = p$. $G' = \langle a^2 \rangle$ is the p -Sylowsubgroup of G with complement $H := \{1, b, a^{p^r}, a^{p^r}b\}$. KH is a radical complement in KG with K -basis H , and the set $\{a^{2s} - 1, b(a^{2s} - 1), a^{p^r}a^{2s} - 1, a^{p^r}ba^{2s} - 1 \mid s \in \underline{p^r - 1}\}$ is a K -basis of the radical. The Cartan-Subalgebras are conjugated by $1 + \text{rad}(KG)$ to $C_{KG}(KH) = C_{\text{rad}(KG)}(KH) \oplus KH$. We calculate the centralizer of KH in $\text{rad}(KG)$ and determine that its dimension is $n - 2$. Hence all Cartan-Subalgebras of $\text{von}(KG)^\circ$ are of dimension $n + 2$.

Let $x \in \text{rad}(KG)$ like

$$\begin{aligned} x &= \sum_{i=1}^{p^r-1} l_i(a^{2i} - 1) + \sum_{i=1}^{p^r-1} m_i b(a^{2i} - 1) \\ &+ \sum_{i=1}^{p^r-1} r_i a^{p^r}(a^{2i} - 1) + \sum_{i=1}^{p^r-1} s_i a^{p^r} b(a^{2i} - 1). \end{aligned}$$

x centralizes KH if and only if $x \circ b = 0 = x \circ a^{p^r}$ is valid.

We calculate

$$\begin{aligned} x \circ a^{p^r} &= \sum_{i=1}^{p^r-1} m_i (ba^{2i}a^{p^r} - a^{p^r}ba^{2i} - ba^{p^r} + a^{p^r}b) \\ &+ \sum_{i=1}^{p^r-1} s_i (a^{p^r}ba^{2i}a^{p^r} - ba^{p^r} - a^{p^r}ba^{p^r} + b) \\ &= 0, \end{aligned}$$

since a^{p^r} is an involution.

Additionally we calculate

$$\begin{aligned} x \circ b &= \sum_{i=1}^{p^r-1} l_i(a^{2i}b - ba^{2i}) + \sum_{i=1}^{p^r-1} m_i(ba^{2i}b - a^{2i}) \\ &+ \sum_{i=1}^{p^r-1} r_i(a^{p^r}a^{2i}b - ba^{p^r}a^{2i} - a^{p^r}b + ba^{p^r}) \\ &+ \sum_{i=1}^{p^r-1} s_i(a^{p^r}ba^{2i}b - ba^{p^r}ba^{2i} - a^{p^r}bb + ba^{p^r}b) \\ &= \sum_{i=1}^{p^r-1} l_i(a^{2i} - a^{-2i})b + \sum_{i=1}^{p^r-1} (-m_i)(a^{2i} - a^{-2i}) \\ &+ \sum_{i=1}^{p^r-1} r_i(a^{2i} - a^{-2i})a^{p^r}b + \sum_{i=1}^{p^r-1} (-s_i)(a^{2i} - a^{-2i})a^{p^r} \\ &= \sum_{i=1}^{\frac{p^r-1}{2}} (l_i - l_{p^r-i})a^{2i}b + \sum_{i=1}^{\frac{p^r-1}{2}} (m_{p^r-i} - m_i)a^{2i} \end{aligned}$$

$$+ \sum_{i=1}^{\frac{p^r-1}{2}} (r_i - r_{p^r-i}) a^{2i} a^{p^r} b + \sum_{i=1}^{\frac{p^r-1}{2}} (s_{p^r-i} - s_i) a^{2i} a^{p^r}.$$

Hence x centralize b if and only if for all $i \in \frac{p^r-1}{2}$ the conditions $l_i = l_{p^r-i}$, $m_i = m_{p^r-i}$, $r_i = r_{p^r-i}$ and $s_i = s_{p^r-i}$ are valid. \diamond

Lie-nilpotent Associative Algebras

As we have deduced by proposition 1 the Cartan-Subalgebras of Lie-Algebras associated to associative algebras are associative subalgebras. Hence we are interested in the associative structure of these special subalgebras. In particular we clarify which associative algebras are Lie-nilpotent. This topic is linked to the question for which unitary associative algebras its group of units is nilpotent. We demonstrate are conclusions for group algebras.

A first result is an easy consequence of theorem 1 in [8]:

Corollary 1

Let A be a finite-dimensional associative unitary K -algebra. A° is nilpotent if and only if every separable element of A is central.

Proof: By Theorem 1 in [8] we conclude that A° is nilpotent if and only if every maximal torus of A is central. If a is a separable element of a then the subalgebra $K[a]$ is a torus (see lemma 5.2.5 in [9]) of A . Hence this torus is contained in a maximal torus of A . \diamond

Remark 4

Let A be a finite-dimensional associative K -algebra with a central radical complement. For all $n \in \mathbb{N}$ the equation $\underbrace{A \circ \cdots \circ A}_{n-mal} = \underbrace{rad(A) \circ \cdots \circ rad(A)}_{n-mal}$

is valid. As $rad(A)$ is a nilpotent associative algebra the Lie-Algebra $rad(A)^\circ$ is nilpotent as well. Hence A° is nilpotent with the same nilpotency class as $rad(A)^\circ$. \diamond

Lemma 2

Every finite-dimensional associative Lie-nilpotent K -algebra is a soluble associative K -algebra.

Proof: Step 1: Let A be a central K -division algebra, $n := ind(D)$ and T be a maximal subfield of A . Then $A \otimes T$ is isomorphic to $T^{n \times n}$. By using

remarks 2 and 3 we conclude $A = K \cdot 1_A$.

Step 2: Let A be a K -division algebra. A is as $Z(A)$ -algebra central and still Lie-nilpotent. By step 1 we get $A = Z(A)$.

Step 3: Let A be simple. Then a K -division algebra D and $n \in \mathbb{N}$ exist such that A is isomorphic to $D^{n \times n}$. As A is Lie-nilpotent so D is Lie-nilpotent, too. By step 2 we conclude that D is a field. In addition A° contains a subalgebra isomorphic to $gl(n, K)$. By remark 3 we get $n = 1$, and hence A is a field.

Step 4: Let A semi-simple and therefor a direct product of simple ideals of A . By step 3 every simple ideal is a field and hence A is commutative.

Step 5: By $A^\circ / rad(A)^\circ = (A / rad(A))^\circ$ and step 4 we get that A is soluble. \diamond

By lemma 2 and proposition 1 we conclude:

Corollary 3

Let A be a finite-dimensional associative unitary K -algebra. Every Cartan-Subalgebra of A° is a soluble associative unitary subalgebra of A . \diamond

A detailed description of the associative structure is given by the next theorem.

Theorem 5

Let A be a finite-dimensional associative unitary K -algebra with separable radical factor algebra. The following statements are equivalent:

- (i) A° is nilpotent.
- (ii) A has a central radical complement.
- (iii) A is soluble and has exactly one radical complement.
- (iv) A is soluble and the set of all separable elements is a radical complement.
- (v) A is soluble and the set of all separable elements is a K -subspace.

Proof: (ii) \rightarrow (i): see remark 4

(i) \rightarrow (ii): Let A° be nilpotent. Using lemma 2 we get that A is soluble. Let T be a radical complement of A . By theorem 4 we get $C_A(T) = A$.

Hence T is central in A . By corollary 2.3.7 in [9] we conclude (ii).

(ii) \rightarrow (iii): As the radical complement is central it is commutative and hence A is soluble. In addition A possesses exactly one radical complement .

(iii) \rightarrow (ii): Let A be soluble and it exists exactly one radical complement of A . By corollary 5.1.5 in [9] the intersection of all radical complements is central.

(iv) \rightarrow (iii): The set of separable elements of A is invariant under conjugation by the group of units. Hence (iii) is a consequence of (iv).

(iii) \rightarrow (iv): Let T be a radical complement of A and A soluble. T is a commutative separable subalgebra of A . By lemma 1 every element of T is separable over K .

Let a be a separable element of A then $K[a]$ is due to lemma 1 a separable subalgebra of A . This subalgebra is – by using corollary 2.3.7 in [9] – contained in a radical complement of A and hence contained in T .

(iv) \rightarrow (v): This is obvious.

(v) \rightarrow (iv): Let T be a radical complement of A . T is commutative and separable. By theorem 5.6.11 in [9] we conclude that every element of T is separable over K . The set of all separable elements is direct to the radical. By dimension reasons we conclude that T is exactly the set of all separable elements of A . \diamond

As a corollary we get a generalization of remark 2:

Corollary 2

Let A, B finite-dimensional associative K -algebras with separable radical factor algebras, A° and B° be nilpotent. Then $(A \otimes B)^\circ$ is nilpotent and $A \otimes B$ has a separable radical factor algebra.

Proof: Let T and S be radical complements of A and B . These complements are central by theorem 5. Theorem 2.2.9 in [9] shows us that $T \otimes S$ is a separable radical complement of $A \otimes B$. As $Z(A \otimes B) = Z(A) \otimes Z(B)$ this radical complement is central and hence $A \otimes B$ is Lie-nilpotent using theorem 5 again. \diamond

Proposition 4

Let A be a finite-dimensional associative unitary K -algebra.

- (i) If $\text{rad}(A)$ has a central complement then $E(A)$ is nilpotent.
- (ii) Is T a radical complement of A then $1_A \in T$.
- (iii) Is T a radical complement of A and $E(T)$ central in A then $E(A)$ is a direct product of the nilpotent normal subgroup $1_A + \text{rad}(A)$ and the central normal subgroup $E(T)$. In particular $E(A)$ is nilpotent.

Proof: see 1.1.8 in [10] and 1.10.1 in [9] \diamond

Lemma 3

Let A be a finite-dimensional associative unitary K -algebra with nilpotent group of units $E(A)$. Then A is soluble.

Proof: By lemma A.1.1 in [9] we get $E(A/\text{rad}(A)) = E(A)/(1 + \text{rad}(A))$ which is nilpotent as well. Let D_1, \dots, D_r be K -division algebras and $n_1, \dots, n_r \in \mathbb{N}$ such that $A/\text{rad}(A)$ is isomorphic to $D_1^{n_1 \times n_1} \times \dots \times D_r^{n_r \times n_r}$. By induction we get from 1.1.8 in [10] that $E(A)/(1 + \text{rad}(A))$ is isomorphic to $E(D_1^{n_1 \times n_1}) \times \dots \times E(D_r^{n_r \times n_r})$. In particular we have that for all $i \in \underline{r}$ the group $E(D_i)$ is nilpotent. By a theorem of Stuth (see Corollary 5.3.1.2 in [5]) we conclude that for all $i \in \underline{r}$ the division algebra D_i is a field. In addition for all $i \in \underline{r}$ the group $GL(n_i, K)$ is nilpotent. Apart from $GL(2, 2)$ and $GL(2, 3)$ these groups (see page 181 in [3]) are not soluble for $n_i \geq 2$. By page 183 in [3] the groups $GL(2, 2)$ and $PSL(2, 3)$ are isomorphic to S_3 and A_4 which are not nilpotent as well. Thus we conclude $n_i = 1$ for all $i \in \underline{r}$. \diamond

Theorem 6

Let K be a field with more than 2 elements and A be an associative finite-dimensional unitary K -algebra with separable radical factor algebra. The following statements are equivalent:

- (i) $E(A)$ is nilpotent.
- (ii) A possesses a central radical complement.
- (iii) A° is nilpotent.

Proof: Due to theorem 5 and proposition 4 we have to prove only the implication (i) \rightarrow (ii). Let $E(A)$ be nilpotent. Then A is soluble by lemma 3. Let T be a radical complement of A (Wedderburn-Malcev). By theorem 5.16

and corollary 5.18 in [1] the group $E(C_A(T))$ is a Carter-Subgroup of $E(A)$. As a Carter-Subgroup is maximal nilpotent we get $E(A) = E(C_A(T))$. In particular we get $1 + \text{rad}(A) \subseteq E(C_A(T)) \subseteq C_A(T)$. Hence $1_A + \text{rad}(A)$ centralizes the radical complement T . By the Wedderburn-Malcev-Theorem we conclude that A possesses exactly one radical complement. Due to corollary 5.1.5 in [9] the intersection of all radical complements of A – which is in our case T – is central. \diamond

Theorem 7

Let A be a finite-dimensional associative unitary K -algebra with separable radical factor algebra. $E(A)$ is nilpotent if and only if for every radical complement T of A the group $E(T)$ is central. In that case $E(A)$ is the direct product of the nilpotent normal subgroup $1 + \text{rad}(A)$ and the central normal subgroup $E(T)$. Sufficient for the nilpotency of $E(A)$ is the nilpotency of A° .

Proof: One implication is the content of proposition 4. Let $E(A)$ be nilpotent. Then A is soluble by lemma 3. For a radical complement T of A the subgroup $C_{E(A)}(E(T))$ is a Carter-Subgroup of $E(A)$ (see theorem 5.16 in [1]). As Carter-Subgroups are maximal nilpotent we conclude $E(A) = C_{E(A)}(E(T))$. Hence $E(T)$ is central. The proof is completed by 1.1.8 in [10] and theorem 5. \diamond

Group Algebras

Let K be a field and G be a finite group. The authors of [2] and [6] prove that KG is Lie-nilpotent if and only if $E(KG)$ is nilpotent. This is equivalent to $\text{char}(K) = 0$ and G is abelian or – in case $\text{char}(K) = p$ – G' is a p -subgroup of the nilpotent group G .

With respect to theorems 6 and 7 we show that KG has a separable radical factor algebra and a central radical complement in the modular case.

Let $\text{char}(K) = p$ and G' be a p -subgroup of the nilpotent group G . Let P be the normal p -Sylowsubgroup of G with normal complement N . By $G' \leq P$ the normal subgroup N is central in G . Let α be the linearization of the canonical epimorphism from G onto the factor group G/P . Then it is well-known that $\text{Kern } \alpha = KG \text{Aug}(KP) = \text{Aug}(KP)KG$ and $\text{Aug}(KP)$ is the augmentation ideal of KP . By a theorem of Wallace $\text{Aug}(KP)$ is nilpotent. As a consequence $\text{Kern } \alpha$ is nilpotent, too. The factor algebra of KG modulo $\text{Kern } \alpha$ is isomorphic to $K(G/P)$ and hence isomorphic to KN . KN is by theorem of Maschke semi-simple and therefore separable by 1.9.4 in [9]. We conclude that $\text{rad}(KG) = KG \text{Aug}(KP)$ holds and that KN is a separable radical complement in KG . As N is central in G the algebra KN is a central radical complement of KG . \diamond

Simple, semi-simple and separable algebras

We analyze Cartan-Subalgebras of Lie-Algebras associated to simple finite-dimensional associative K -algebras and reduce the analysis of semi-simple (and separable) to their simple components. The following lemma is a version of the corresponding one from S. Siciliano in [8]:

Lemma 4

Let A be a central-simple finite-dimensional associative K -algebra. Is H a Cartan-Subalgebra of A° then H is a $\text{ind}(A)$ -dimensional self-centralizing torus of A . In particular H is a maximal commutative subalgebra of A .

Proof: By proposition 1 and lemma 2 the algebra H is a soluble unitary subalgebra of A . Let $n := \text{ind}(A)$. For a maximal subfield T of A the tensor product $A \otimes T$ is isomorphic to $T^{n \times n}$. Is F an algebraic closure of T then $A \otimes F \cong F^{n \times n}$ holds, and we conduct $n^2 = \dim_K(A) = \dim_F(A \otimes F)$. By [4] the algebra $H \otimes F$ is a Cartan-Subalgebra of the F -algebra $(A \otimes F)^\circ$ isomorphic to $\mathfrak{gl}(n, F)$. Again by [4] all Cartan-Subalgebras of $(F^{n \times n})^\circ$ are conjugate under $GL(n, F)$ to the diagonal-matrix-algebra $D(n, F)$. Hence every element of the F -algebra $H \otimes F$ is diagonalizable and therefor every element of H is separable over K . In particular H is semi-simple. As H is soluble (lemma 2) H is a torus. H is self-centralizing as H is commutative and self-normalizing: $H \subseteq C_A(H) = C_{A^\circ}(H) \subseteq N_{A^\circ}(H) = H$. Finally for each commutative subalgebra C of A that contains H the statement $C \subseteq C_A(C) \subseteq C_A(H) = H$ is valid. \diamond

Theorem 8

Let A be a central-simple finite-dimensional associative K -algebra. The following statements are valid:

- (i) The maximal tori of A are exactly the self-centralizing tori of A . In particular, every maximal torus of A is a maximal commutative, separable subalgebra of A .
- (ii) The Cartan-Subalgebras of A° are exactly the maximal tori of A .
- (iii) Every Cartan-Subalgebra of A° is $\text{ind}(A)$ -dimensional.
- (iv) All Cartan-Subalgebras of A° are isomorphic.

Proof: ad(i): Let T be a maximal torus of A . Using theorem 1 in [8] the subalgebra $C_A(T)$ is a Cartan-Subalgebra of A° which is by lemma 4 a torus as well. As T is commutative we conclude $T = C_A(T)$. For a commutative subalgebra C of A which contains T we deduct: $C \subseteq C_A(C) \subseteq C_A(T) = T$.

Finally T is separable by lemma 1.

ad(ii): This statement is a consequence of (i) and theorem 1 in [8].

ad(iii): see lemma 4.

ad(iv): All Cartan-Subalgebras of A° are abelian by (ii) and by (iii) of the same K -dimension. \diamond

Every simple finite-dimensional associative algebra A is central as $Z(A)$ -algebra. Every Cartan-Subalgebra of A° contains the center of A and is therefor by proposition 1 a $Z(A)$ -subalgebra of A . Hence we conclude by theorem 8:

Theorem 9

Let A be a simple finite-dimensional associative K -algebra.

The Cartan-Subalgebras of A° are exactly those unitary commutative subalgebras of A which are maximal with respect to that every element is separable over the center of A . These subalgebras are self-centralizing and maximal commutative.

In particular each Cartan-Subalgebra T of A° is a direct sum of fields and $\text{ind}_{Z(A)}(A)$ -dimensional. All Cartan-Subalgebras of A° are isomorphic. \diamond

By theorems 8 and 9 we conclude:

Corollary 3

Let A be a simple finite-dimensional associative K -algebra for which the center is separable over K .

- (i) The maximal tori are exactly the self-centralizing tori of A . In particular every maximal torus of A is a maximal commutative, separable subalgebra von A .
- (ii) The Cartan-Subalgebras of A° are exactly the maximal tori of A .
- (iii) Every Cartan-Subalgebra T of A° is $\text{ind}_{Z(A)}(A)$ -dimensional.
- (iv) The Cartan-Subalgebras of A° are isomorphic. \diamond

Remark 5

We will reduce now the analysis of Cartan-Subalgebras for semi-simple associative algebras to their simple components.

Let A, B be associative K -algebras. For all $a_1, a_2 \in A$ and $b_1, b_2 \in B$ the

rule $(a_1; b_1) \circ (a_2; b_2) = (a_1 \circ a_2; b_1 \circ b_2)$ is valid. Hence we conclude for $T \subseteq A$ and $S \subseteq B$ the equation $N_{(A \times B)^\circ}(T \times S) = N_{A^\circ}(T) \times N_{B^\circ}(S)$.

(i) Let $n \in \mathbb{N}$, A_1, \dots, A_n be associative K -algebras and C_1, \dots, C_n Cartan-Subalgebras of $(A_1)^\circ, \dots, (A_n)^\circ$ then $C_1 \times \dots \times C_n$ is a Cartan-Subalgebra of $(A_1 \times \dots \times A_n)^\circ$.

(ii) Let A, B be associative finite-dimensional K -algebras and C be a Cartan-Subalgebra of $(A \times B)^\circ$. We define $T := \{a \mid a \in A, \exists b \in B : (a, b) \in C\}$ and $S := \{b \mid b \in B, \exists a \in A : (a, b) \in C\}$. T resp. S is a nilpotent subalgebra of A° resp. B° . In particular $T \times S$ is a nilpotent subalgebra of $(A \times B)^\circ$ containing (by definition) C as a subalgebra. As C is maximal nilpotent we conclude $C = T \times S$. By $T \times S = C = N_{(A \times B)^\circ}(C) = N_{(A \times B)^\circ}(S \times T) = N_{A^\circ}(T) \times N_{B^\circ}(S)$ we deduct finally that T resp. S is a Cartan-Subalgebra of A° resp. B° . \diamond

We deduct the following reduction-theorem which includes the description of Cartan-Subalgebras in the semi-simple case:

Theorem 10

Let $n \in \mathbb{N}$ and A_1, \dots, A_n be finite-dimensional associative K -algebras. The Cartan-Subalgebras of $(A_1 \times \dots \times A_n)^\circ$ are exactly the subalgebras $C_1 \times \dots \times C_n$, whereas for every $i \in \underline{n}$ the set C_i is a Cartan-Subalgebra of $(A_i)^\circ$. \diamond

In particular we conclude from corollary 3, theorems 10 and theorem 1 in [8]:

Theorem 11

Let A be a finite-dimensional associative separable K -algebra.

- (i) Every maximal torus of A is a direct sum of maximal tori linked to the direct composition of A into simple ideals of A .
- (ii) The maximal tori of A are exactly the self-centralizing tori of A . In particular every maximal torus is a maximal commutative, separable subalgebra of A .
- (iii) The Cartan-Subalgebras of A° are exactly the maximal tori of A .
- (iv) All Cartan-Subalgebras of A° are isomorphic. \diamond

Reduced Algebras

For an associative K -algebra A we denote by $nil(A)$ the set of all nilpotent elements of A . The nil radical of A is always a subset of $nil(A)$.

Definition 2

An associative K -algebra is called reduced if and only if $rad(A) = nil(A)$ is valid.

We will reduce the analysis of Cartan-Subalgebras of reduced associative algebras to some special soluble subalgebras.

An easy observation of reduced algebras is given in the following proposition:

Proposition 5

Let A be a finite-dimensional associative K -algebra. The following statements are equivalent:

- (i) A is reduced.
- (ii) $A/rad(A)$ is reduced.
- (iii) $A/rad(A)$ is a direct sum of K -division algebras.
In particular A is reduced when A is commutative.
- (iv) For every subalgebra T of A the condition $rad(T) = rad(A) \cap T$ is valid.
- (v) Every subalgebra of A is reduced. \diamond

By using lemma 1 the following remark is valid:

Remark 6

Let A be a finite-dimensional associative unitary K -algebra, T a torus of A and $S := T \oplus rad(A)$.

Then T is a separable complement of the radical $rad(S) = rad(A)$ of S and S a soluble subalgebra of A . \diamond

Lemma 5

Let A be a associative finite-dimensional unitary reduced K -algebra.

The maximal elements of the set of soluble subalgebras with separable radical factor algebra are exactly of the form $rad(A) \oplus T$, whereas T is a maximal torus of A .

Proof: Let T be a maximal torus of A and $S := \text{rad}(A) \oplus T$. By remark 6 the set S is a soluble subalgebra of A with separable radical factor algebra. Let B be a soluble subalgebra of A with separable radical factor algebra which contains $\text{rad}(A) \oplus T$. As A is reduced we conclude by proposition 5 the condition $\text{rad}(B) \subseteq \text{rad}(A)$. As a torus T is by lemma 1 a separable subalgebra of B which is by corollary 2.3.7 in [9] contained in a radical complement X of B . X is using lemma 1 a torus of A as well, and we deduct $T = X$ from the maximality of T . Therefor we proved $B = \text{rad}(A) \oplus T$. For the other implication let B be a maximal element of the set of soluble subalgebras with separable radical factor algebra and T be a radical complement of B . The subalgebra T is by lemma 1 a torus of A , and by proposition 5 we conclude $\text{rad}(B) \subseteq \text{rad}(A)$. We define $S := \text{rad}(A) \oplus T$. Using remark 6 and the maximality of B we conclude $B = S$ and $\text{rad}(B) = \text{rad}(A)$. Suppose R is a T containing torus not equal to T then the subalgebra $\text{rad}(A) \oplus R$ is containing B and not equal to B . With remark 6 we deduct a contradiction to the maximality of B . \diamond

By lemma 5 we get the following corollary:

Corollary 4

Let A be a associative finite-dimensional unitary reduced K -algebra over a perfect field K . The maximal soluble subalgebras of A are exactly of the form $T \oplus \text{rad}(A)$ whereas T is a maximal torus of A . \diamond

Remark 7

Let A be a associative K -algebra, I an ideal and T a subalgebra of A such that A is the internal direct sum of I and T . For every subset X of T the equation $C_A(X) = C_I(X) \oplus C_T(X)$ is valid. \diamond

Theorem 12

Let A be a finite-dimensional associative unitary reduced K -algebra with separable radical factor algebra and $\mathcal{S}(A)$ the set of soluble subalgebras with separable radical factor algebra of A .

The Cartan-Subalgebras of A° are exactly the Cartan-Subalgebras of Lie-Algebras associated to maximal elements of $\mathcal{S}(A)$.

Proof: Let H be a Cartan-Subalgebra of A° . By theorem 1 in [8] there exists a maximal torus T of A such that $H = C_A(T)$ is valid. We define $S := \text{rad}(A) \oplus T$. By lemma 5 the set S is a maximal element of $\mathcal{S}(A)$. Using 2.3.7 in [9] and lemma 1 there exists a radical complement C of A containing T . Thus T is a maximal torus of C which is by theorem 11 in C self-centralizing. By remark 6 we conclude $H = C_A(T) = C_{\text{rad}(A)}(T) \oplus T$. T

is of course a maximal torus of S . By theorem 1 in [8] the subalgebra $C_S(T)$ is a Cartan-Subalgebra of S° . As T is commutative we get by remark 7 the statement $C_S(T) = C_{rad(A)}(T) \oplus T = C_A(T) = H$. Hence H is a Cartan-Subalgebra von S .

Conversely let S be a maximal element of $\mathcal{S}(A)$. By lemma 5 there exists a maximal torus T of A such that $S = rad(A) \oplus T$ is valid. Every Cartan-Subalgebra of S is due to theorem 4 the centralizer of a radical complement of S . These are by the Wedderburn-Malcev-Theorem conjugates of T with respect to $1 + rad(S) = 1 + rad(A)$. Now let H be a Cartan-Subalgebra of S . We can assume $H = C_S(T)$. By remark 6 and the commutativity of T we deduct $H = C_S(T) = C_{rad(A)}(T) \oplus T$. Using theorem 1 in [8] the subalgebra $C_A(T)$ is a Cartan-Subalgebra of A° . Let C be a radical complement of A containing T (see 2.3.7 in [9] and lemma 1). By remark 6 and theorem 11 we conclude $C_A(T) = C_{rad(A)}(T) \oplus C_C(T) = C_{rad(A)}(T) \oplus T = C_S(T) = H$. \diamond

By theorem 12 we conclude the following corollary:

Corollary 5

Let A be finite-dimensional associative unitary reduced K -algebra over a perfect field K . The Cartan-Subalgebras of A° are exactly the Cartan-Subalgebras of Lie-Algebras associated to maximal soluble subalgebras of A . \diamond

Associative Algebras

Theorem 13

Let A be a associative finite-dimensional unitary K -algebra with separable radical factor algebra and H be a subset of A . The following statements are equivalent:

- (i) H is a Cartan-Subalgebra of A° .
- (ii) It exists a radical complement C of A and a Cartan-Subalgebra T of C° such that $H = C_A(T)$.

Proof: (i) \rightarrow (ii): Let H be a Cartan-Subalgebra of A° . By theorem 1 in [8] there exists a maximal torus T of A such that $H = C_A(T)$ is valid. By 2.3.7 in [9] and lemma 1 the torus T is contained in a radical complement C of A . T is a maximal torus of C as well. By theorem 11 the separable subalgebra T is a Cartan-Subalgebra of C° .

(ii) \rightarrow (i): Let C be a radical complement of A and T be a Cartan-Subalgebra of C° . By theorem 11 the subalgebra T is a maximal torus of C such that

$T = C_C(T)$. By remark 6 we conclude $C_A(T) = C_C(T) \oplus C_{rad(A)}(T) = T \oplus C_{rad(A)}(T)$. Obviously T is a central radical complement in $C_A(T)$. Using remark 4 we deduct that $C_A(T)$ is Lie-nilpotent. T is contained in a maximal torus S of A . By theorem 1 in [8] the subalgebra $C_A(S)$ is a Cartan-Subalgebra of A° . $C_A(S)$ is maximal nilpotent and contained in $C_A(T)$. Hence we get $C_A(T) = C_A(S)$ and $C_A(T)$ is a Cartan-Subalgebra of A° . \diamond

By theorems 11 and 13 we conclude the following **strategy** for calculating a Cartan-Subalgebra of Lie-Algebras associated to associative unitary finite-dimensional algebras with separable radical factor algebra:

(1) Determine a maximal tori of the radical complements. These are exactly the self-centralizing tori of the radical complements. To determine a self-centralizing torus in a radical complement C begin with a (as big as possible) torus T . Calculate the centralizer of the torus in C and search for a separable element t of this centralizer not contained in T . The subalgebra $K[T, t]$ is a torus as well which contains T . Repeat this approach until no such separable element can be determined.

(2) For determining a Cartan-Subalgebras of A° calculate for a maximal torus of (1) the centralizer in A (theorem 13). This centralizer is by remark 6 and theorem 11 exactly $C_{rad(A)}(T) \oplus T$.

This approach is demonstrated now by giving an alternative proof of theorem 4 and by determining a Cartan-Subalgebra of $(KD_{2n})^\circ$.

Corollary 5

Let A be a finite-dimensional associative unitary soluble K -algebra with separable radical factor algebra. The Cartan-Subalgebras of A° are exactly the centralizers of the radical complements of A .

Proof: Let C be a radical complement of A . By using lemma 1 the subset C is a torus. The only maximal torus of C is C itself. \diamond

Group algebras of Dihedral Groups

Let $n \in \mathbb{N}$, $G := D_{2n}$ a Dihedral Group and K be a field with $char(K) = p$. There exists $a, b \in G$ such that $o(a) = n$, $o(b) = 2$, $G = \langle a, b \rangle$, $a^b = a^{-1}$ and $G = \{1, a, \dots, a^{n-1}, b, ab, \dots, a^{n-1}b\}$ are valid. Is $p > 0$ not a factor of n or $p = 0$ then – by a theorem of Maschke – the subalgebra $K\langle a \rangle$ is semi-simple (and commutative). By 1.9.4.2 in [9] the algebra $K\langle a \rangle$ is separable over K , and by lemma 1 we conclude that $K\langle a \rangle$ is a torus of KG . We analyze

the centralizer of $K\langle a \rangle$ in KG . We have $KG = K\langle a \rangle \oplus \langle b, ab, \dots, a^{n-1}b \rangle_K$, and $K\langle a \rangle$ centralizes $K\langle a \rangle$. Let $x \in \langle b, ab, \dots, a^{n-1}b \rangle_K$, like $x = \sum_{i=0}^{n-1} k_i a^i b$. x centralizes $K\langle a \rangle$ if and only if $\sum_{i=0}^{n-1} k_i a^i b^a = \sum_{i=0}^{n-1} k_i a^i b$ is valid. Because of $b^a = a^{-2}b$ this is equivalent to $a^{-2}(\sum_{i=0}^{n-1} k_i a^i) = \sum_{i=0}^{n-1} k_i a^i$. If 2 is not a factor of n (hence $\langle a^{-2} \rangle = \langle a \rangle$ is valid) then x centralizes $K\langle a \rangle$ if and only if $K\langle a \rangle$ acts trivial on $\sum_{i=0}^{n-1} k_i a^i \in K\langle a \rangle$. This is equivalent to $\sum_{i=0}^{n-1} k_i a^i \in \langle \sum_{i=0}^{n-1} a^i \rangle_K$. We conclude $C_{KG}(K\langle a \rangle) = K\langle a \rangle \oplus \langle b + ab + \dots + a^{n-1}b \rangle_K = K\langle a \rangle \oplus \langle \sum_{g \in G} g \rangle_K$.

Is p a factor of the order of G – hence we have $p = 2$ under our conditions – then the one-dimensional K -subspace $\langle \sum_{g \in G} g \rangle_K$ is a zero-ideal contained in nil radical of KG . We assume that $KG/\text{rad}(KG)$ is separable. The torus $K\langle a \rangle$ is contained by lemma 1 and 2.3.7 in [9] in a radical complement C of KG . By remark 6 we have $C_{KG}(K\langle a \rangle) = C_{\text{rad}(KG)}(K\langle a \rangle) \oplus C_C(K\langle a \rangle)$. In addition the statements $C_{KG}(K\langle a \rangle) = K\langle a \rangle \oplus \langle \sum_{g \in G} g \rangle_K$, $K\langle a \rangle \subseteq C_C(K\langle a \rangle)$ and $\langle \sum_{g \in G} g \rangle_K \subseteq C_{\text{rad}(KG)}(K\langle a \rangle)$ are valid. Thus $K\langle a \rangle$ self-centralizing in C . By the theorems 11 and 13 the set $C_{KG}(K\langle a \rangle) = K\langle a \rangle \oplus \langle \sum_{g \in G} g \rangle_K$ is a $(n+1)$ -dimensional Cartan-Subalgebra of $(KG)^\circ$.

Let p not be a factor of the order of the group G . The element $s := \sum_{g \in G} g$ is diagonalizable (and hence separable over K) because of $s^2 = |G| \cdot s$. Using our strategy $C_{KG}(K\langle a \rangle)$ is a torus of A which is obviously self-centralizing in KG . Hence $C_{KG}(K\langle a \rangle)$ is again a Cartan-Subalgebra of $(KG)^\circ$. \diamond

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